



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE ENVELOPE OF THE AXES OF A SYSTEM OF CONICS PASSING THROUGH THREE FIXED POINTS*

BY

R. E. ALLARDICE

In a recent number of the *Annals of Mathematics*† I have shown that the envelope of the asymptotes of a system of conics passing through three fixed points consists of two three-cusped hypocycloids, touching the three straight lines that join the three fixed points in pairs. I propose now to show that the envelope of the axes of the same system of conics consists of two three-cusped hypocycloids touching three concurrent straight lines.

The foci of a conic may be regarded as four of the vertices of a complete four-side circumscribing the conic, the other two vertices being the circular points at infinity; then the straight line at infinity is one diagonal line of this four-side, and the axes are the other two diagonal lines.

The coördinates of the circular points at infinity are $(1, -e^{C\alpha}, -e^{-B\alpha})$ and $(1, -e^{-C\alpha}, -e^{B\alpha})$; let us denote these points for the present by (x_1, y_1, z_1) and (x_2, y_2, z_2) .

Let the equation of the conic be

$$U \equiv \lambda_1 yz + \lambda_2 zx + \lambda_3 xy = 0,$$

the three fixed points through which the conic is to pass being the vertices of the triangle of reference; and put

$$U_1 \equiv \lambda_1 y_1 z_1 + \lambda_2 z_1 x_1 + \lambda_3 x_1 y_1; \quad U_2 \equiv \lambda_1 y_2 z_2 + \lambda_2 z_2 x_2 + \lambda_3 x_2 y_2;$$

$$U'_1 \equiv x_1 \frac{\partial U}{\partial x} + y_1 \frac{\partial U}{\partial y} + z_1 \frac{\partial U}{\partial z}; \quad U'_2 \equiv x_2 \frac{\partial U}{\partial x} + y_2 \frac{\partial U}{\partial y} + z_2 \frac{\partial U}{\partial z}.$$

* Presented to the Society (Chicago) January 2, 1903. Received for publication August 2, 1902.

† *On some curves connected with a system of similar conics*, *Annals of Mathematics*, 2d series, vol. 3 (1902), p. 154.

The equations of the tangents from the circular points at infinity are

$$U_1'^2 = 4UU_1$$

and

$$U_2'^2 = 4UU_2;$$

and the foci are the intersections of these two pairs of straight lines.

The equation $U_2U_1'^2 = U_1U_2'^2$ evidently represents a third pair of straight lines passing through the foci, and must therefore represent the axes.

Now the condition for similarity may be expressed in the form *

$$\sum (\lambda_1^2 \sin^2 A - 2\lambda_2\lambda_3 \sin B \sin C) = t^2 (\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2,$$

where t is the tangent of the angle between the asymptotes; or,

$$\sum (\lambda_1^2 - 2\lambda_2\lambda_3 \cos A) = s^2 (\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2,$$

that is,

$$U_1U_2 = s^2P^2,$$

where s is the secant of the angle between the asymptotes, and

$$P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C.$$

Hence the equations of the axes may be expressed in the form

$$U_1U_2' = sPU_1', \quad \text{or} \quad U_2U_1' = sPU_2';$$

and

$$U_1U_2' = -sPU_1', \quad \text{or} \quad U_2U_1' = -sPU_2'.$$

Using the first of these equations, we may write the tangential coördinates of the corresponding axis in the form

$$u = U_1(\lambda_2z_2 + \lambda_3y_2) - sP(\lambda_2z_1 + \lambda_3y_1),$$

$$v = U_1(\lambda_3x_2 + \lambda_1z_2) - sP(\lambda_3x_1 + \lambda_1z_1),$$

$$w = U_1(\lambda_1y_2 + \lambda_2x_2) - sP(\lambda_1y_1 + \lambda_2x_1).$$

Noticing that

$$\lambda_1(y_1z_2 + y_2z_1) + \lambda_2(z_1x_2 + z_2x_1) + \lambda_3(x_1y_2 + x_2y_1) = -2P,$$

we have, on multiplying these equations first by x_1, y_1, z_1 and adding, and then by x_2, y_2, z_2 , and adding,

$$V \equiv x_1u + y_1v + z_1w = -2PU_1 - 2sPU_1,$$

$$W \equiv x_2u + y_2v + z_2w = 2U_1U_2 + 2sP^2.$$

*See the paper entitled, *On some curves, etc.*, referred to above.

Hence, taking account of the relation $U_1 U_2 = s^2 P^2$, we find,

$$P = \frac{\sqrt{W}}{\sqrt{2s(s+1)}}, \quad U_1 = -\frac{V\sqrt{s}}{\sqrt{2(s+1)W}}.$$

Now writing the coördinates in the form

$$u = (z_2 U_1 - z_1 s P) \lambda_2 + (y_2 U_1 - y_1 s P) \lambda_3,$$

$$v = (z_2 U_1 - z_1 s P) \lambda_1 + (x_2 U_1 - x_1 s P) \lambda_3,$$

$$w = (y_2 U_1 - y_1 s P) \lambda_1 + (x_2 U_1 - x_1 s P) \lambda_2,$$

and using the equation

$$P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C,$$

we have, on eliminating $\lambda_1, \lambda_2, \lambda_3$ the relation

$$\begin{vmatrix} u & 0 & z_2 U_1 - z_1 s P & y_2 U_1 - y_1 s P \\ v & z_2 U_1 - z_1 s P & 0 & x_2 U_1 - x_1 s P \\ w & y_2 U_1 - y_1 s P & x_2 U_1 - x_1 s P & 0 \\ P & \cos A & \cos B & \cos C \end{vmatrix} = 0.$$

On substituting the values of P and U_1 found above, and reducing by means of the relations

$$x_2 V + x_1 W = 2(u - v \cos C - w \cos B), \text{ etc.,}$$

we finally obtain the equation of the envelope in the form

$$\begin{vmatrix} u & 0 & u \cos B + v \cos A - w & w \cos A + u \cos C - v \\ v & u \cos B + v \cos A - w & 0 & v \cos C + w \cos B - u \\ w & w \cos A + u \cos C - v & v \cos C + w \cos B - u & 0 \\ 1/(s+1) & \cos A & \cos B & \cos C \end{vmatrix} = 0,$$

or

$$\begin{aligned} & \sum [u(v \cos C + w \cos B - u) \{u \cos(B - C) - v \cos B - w \cos C\}] \\ & - \frac{2}{s+1} (v \cos C + w \cos B - u)(w \cos A + u \cos C - v)(u \cos B + v \cos A - w) = 0. \end{aligned}$$

It may be shown that this curve has the straight line at infinity for a double tangent, the circular points at infinity being the points of contact.

It must therefore be of the fourth order and have three cusps; and hence for all values of s (except $s = -1$) it is a three-cusped hypocycloid.

It may easily be shown that it always touches the perpendicular bisectors of the sides of the triangle of reference; in the special case, $s = -1$, the curve degenerates into the points at infinity on these three lines.

The two axes envelope the same curve only in the case of the equilateral hyperbola, for which $s = \infty$.

STANFORD UNIVERSITY, CALIFORNIA.
